ON SOME USEFUL CHEMICAL BALANCE WEIGHING DESIGNS FROM FAMILY (A) BALANCED INCOMPLETE BLOCK (BIB) DESIGNS

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1. Introduction

The results of n weighing operations to determine the weight of p light objects on a chemical balance fit into the linear model

$$\underline{Y}=\underset{\sim}{X}\,\underline{W}+\underline{e}$$

where $\underline{Y}$ is the observation vector, $\underset{\sim}{X}=(X_{ij})$ is a matrix of elements -1, 1, 0, $\underline{W}$ is the vector of unknown weights and $\underline{e}$ is the residual vector with $E(\underline{e})=\underline{0}$, $D(\underline{e})=\sigma^2\underset{\sim}{I}$.

Assuming $\underset{\sim}{X}'\underset{\sim}{X}$ to be non-singular the least square estimates are given by,

$\hat{\underline{W}}=(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underline{Y}$, and the dispersion matrix of $\hat{\underline{W}}$ is

$\sigma^2(\underset{\sim}{X}'\underset{\sim}{X})^{-1}$

The weighing design $\underset{\sim}{X}$ is optimum if it estimates each of the weights with variance σ^2/n. We shall introduce a more general definition of an optimum design. Consider a design $\underset{\sim}{X}$ for weighing p objects in n weighing operations where the ith object is included in the weighing operation n_i times ($i=1,\ldots,p$). Then the design $\underset{\sim}{X}$ is optimum for the above mentioned scheme if the weight of the ith object is estimated with variance σ^2/n_i ($i=1,\ldots,p$). Hereafter we shall use the term 'optimum' in the latter sense.

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The question raised here is "How to obtain chemical balance designs when one is interested in more precise estimates on a fraction of objects to be weighed?" A technique has been suggested for combining two family (A) BIB designs [*i.e.* BIB designs for which $b=4\ (r-\lambda)$] which can be used to obtain optimum estimates with at most three types of variance. A large number of useful designs suitable to one's need may be constructed using this technique. The method is easily generalized to the case when one can combine a number of family (A) BIB designs.

2. THE METHOD

Let there be a BIB design with the parameters v, b, r, k, λ Let $\underset{\sim}{N}$ denote the incidence matrix (of order $b \times v$) of this design having the elements 0 and 1. Let us replace the element zero in the above matrix by -1 wherever zero occurs. Call this matrix $\underset{\sim}{X}$. It has been shown by Dey [1] that the matrix $\underset{\sim}{X}$ corresponding to the family of BIB designs satisfying $b=4\ (r-\lambda)$ provide arrangements for optimum chemical balance weighing designs.

Let $\underset{\sim}{N}_1$ and $\underset{\sim}{N}_2$ denote respectively the incidence matrices of the following two family (A) BIB designs :

$$I. \quad v_1, b_1, r_1, k_1, \lambda_1$$

and

$$II. \quad v_2, b_2, r_2, k_2, \lambda_2$$

Let n_1 of the symbols in the two BIB designs be common. The symbols can always be arranged in such a way that first n_1 of them are common in the two designs. Let $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ be the matrices obtained from the incidence matrices of these designs by replacing 0 by -1. $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ can be partitioned as,

$$\underset{\sim}{X}_1=[\underset{\sim}{A}_1 \ \vdots \ \underset{\sim}{B}_1]$$

$$\underset{\sim}{X}_2=[\underset{\sim}{A}_2 \ \vdots \ \underset{\sim}{B}_2]$$

$\underset{\sim}{A}_1$ is of order $b_1 \times n_1$, $\underset{\sim}{B}_1$ of $b_1 \times (v_1-n_1)$, $\underset{\sim}{A}_2$ of $b_2 \times n_1$ and $\underset{\sim}{B}_2$ of order $b_2 \times (v_2-n_1)$.

Define $\underset{\sim}{X}_1^*$ and $\underset{\sim}{X}_2^*$ as

$$\underset{\sim}{X}_1^* = [\,\underset{\sim}{A}_1 : \underset{\sim}{B}_1 : \underset{\sim}{O}_1]$$

$$\underset{\sim}{X}_2^* = [\underset{\sim}{A}_2 : \underset{\sim}{O}_2 : \underset{\sim}{B}_2]$$

The matrix $\underset{\sim}{O}$ consists of zeros only. The orders of $\underset{\sim}{O}_1$ and $\underset{\sim}{O}_2$ are $b_1 \times (v_2-n_1)$ and $b_2 \times (v_1-n_1)$ respectively. Then the matrix X

defined by

$$\underset{\sim}{X}=\begin{bmatrix}\underset{\sim}{X}_1^* \\ \cdots\cdots \\ \underset{\sim}{X}_2^*\end{bmatrix}$$

gives an optimum chemical balance weighing design for weighing $(v_1+v_2-n_1)$ objects in $b=b_1+b_2$ weighing operations. It can be easily seen that $\underset{\sim}{X}'\,\underset{\sim}{X}$ is diagonal.

The weights are estimated with variances

σ^2/b for objects in group I

σ^2/b_1 for objects in group II and

σ^2/b_2 for objects in group III

where group I consists of common objects in the two designs, group II of the remaining (v_1-n_1) objects in design I and group III of the remaining (v_2-n_1) objects in design II.

Thus, this technique gives optimum estimates of all the weights. As indicated earlier, the technique is particularly useful in cases when more accuracy is desired on a fraction of weights to be estimated.

Corollary : When $V_1=n_1$, *i.e.* all the objects in design I are common to design then,

$V(\hat{w}_i)=\sigma^2/b$ for objects in design I and

$=\sigma^2/b_2$ for the remaining objects.

SUMMARY

A technique for obtaining chemical balance weighing designs from family (A) BIB designs is given. The designs obtained are optimum in view of the new definition of an optimum design. The technique is particularly useful in cases when more accuracy is desired on a fraction of weights to be estimated.

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REFERENCE

[1] Dey, A (1971). : On some chemical balance weighing designs. *Austral. J. Statis.*, *13*, 137-141.